# Mathematical Freedom, Truth, and Coherence

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Draft version: please don't cite

**Abstract**

A striking fact about mathematics is that it is, in a certain sense, free or unconstrained in its practice: mathematicians are free, in their capacity as mathematicians, to work within any theory provided that minimal standards of logical coherence are satisfied. How can this feature of mathematics be explained? I argue that contemporary views in the philosophy of mathematics are unable to provide a satisfying explanation. I present a novel account of mathematical truth, drawing on the doctrine of alethic pluralism, according to which (to a first approximation) mathematical truth is realized by the property of coherence. I conclude by showing that this view provides a surprisingly fruitful framework for resolving resolving issues of mathematical disagreement and the question of new set theoretic axioms.

## 1. Mathematical Freedom

There is a long tradition of thought in the history and philosophy of mathematics according to which the subject is, in a certain sense to be discussed, *free* or *unconstrained* in its practice. These words might at first appear to be strange ones to use to describe mathematics. Of all the domains of human inquiry, it is the one most closely associated with the ideal of formal rigour: the requirement that its results be not merely made plausible, but *proven* in a logically unimpeachable way. Against a tacitly understood background for mathematical inquiry – a set of axioms, in the usual case – mathematicians are certainly not free to go around making whatever claims they like! Rather, a proof from those axioms is required; and although the notion of proof employed here is not exactly the same as the maximally explicit one used in logic textbooks, it is nevertheless closely related, and arguably no less rigorous.[1] So mathematics is certainly

[1] The point is clearly and succinctly put by Burgess [2015, p.88]: "none of the logic books' approaches to deduction closely matches the practice of mathematicians in giving what they consider rigorous deductive proofs...". Nevertheless: "If there is something that counts as a proof by ordinary mathematical standards, logicians are confident that there could in principle be written down something that would count as a proof by ideal logical standards."

not unconstrained in its *methodology*. The kind of freedom I want to discuss is different. One aspect of it is expressed by the following thesis:

**Sociological Freedom:** Mathematicians are free, in their capacity as mathematicians, to work within and draw out the consequences of *any* theory, *any* set of axioms they care to, as long as those theories meet minimal standards of logical coherence.

A few clarifications are worth making immediately, before looking at the plausibility of the thesis. Firstly, what exactly does it mean to say that mathematicians are "free" to perform some activity in their capacity as mathematicians? I understand such claims as descriptive claims concerning norms of disciplinary demarcation. The idea is simple: there are standards or norms prevalent in the mathematical community whereby certain investigations or activities count as *doing legitimate mathematics,* and others do not. Sociological Freedom thus expresses the idea that the investigation of any minimally logically coherent set of axioms is regarded, by the internal standards of mathematics, as a legitimate mathematical activity. A second clarification will pre-empt some immediate objections. It is important to keep in mind the difference between, on the one hand, the distinction above between activities that are *legitimate mathematics* and those that are not, and on the other hand, the distinction between *good* (e.g. elegant, important, beautiful, interesting, fruitful, explanatory, unificatory, etc) mathematics and *bad* mathematics. I take Sociological Freedom as a thesis involving the first distinction, not the second: I do not wish to claim that the study of any old axiomatic theory or putative objects will result in *good* mathematics, for that is manifestly false – there are many possible uninteresting, unfruitful, or unimportant theories that could have been studied if not for the better judgement of mathematicians. A last clarification concerns the "minimal standards of logical coherence" in which the thesis is couched. At the start of this inquiry, I'll be somewhat schematic about how we are going to understand the relevant standards; until Section 3, where I will clarify my stance on the issue, I will slide somewhat loosely between a number of closely related notions: proof-theoretic consistency, model-theoretic or semantic satisfiability, logical possibility, and coherence.

Turning back to the thesis itself, one source of support can be seen by considering the trajectory of the history of mathematics, and in particular the striking tendency of disputes concerning the legitimacy of the study of various mathematical theories or putative entities to be resolved in favour of the more permissive side of the dispute. This is not a paper about the history of mathematics. But one (perhaps somehwat "Whiggish") reading of the history of the subject – a reading I find plausible – sees, as an important strand, something like Sociological Freedom emerging as a result of the periodic controversies over the legitimacy of working within theories of various kinds. I can only sketch the case here very briefly. But perhaps the best-known family of examples here concerns the introduction and eventual acceptance of various number-systems: for instance, the acceptance of negative numbers, irrational numbers, complex numbers, and even Hamilton's quaternions, all over the objections of passionate sceptical resistance at the time.[2]

Other historical examples illustrate the same trend. Consider the introduction of non-Euclidean geometries by Bolyai and Lobachevsky – theories which entertain the rejection of Euclid's parallel postulate, and which precipitated great controversy against the background Kantian picture of geometry as a primary example of the synthetic a priori. Or take the disputes surrounding the notions of "function" and "curve", which centered around the legitimacy of discontinuous and non-definable curves. A final example worth mentioning is that of set theory as formulated by Cantor, which was famously dismissed by Kronecker as theology. Again, in each case, the more permissive conception of mathematics emerged victorious.[3]

These historical developments did not go unnoticed – indeed, an explicit recognition of the freedom of mathematicians appears to have been in the air in the late nineteenth century. As Cantor expressed the lessons of the development of mathematics:

> Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. In particular, in the introduction of new numbers it is only obligated to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to the older numbers that they can in any given instance be precisely distinguished. As soon as a number satisfies all these conditions it can and must be regarded in mathematics as existent and real.[4]

To be sure, these claims go beyond Sociological Freedom as I have formulated it above; it seems additionally to express an *alethic* (or *ontological*) thesis concerning the *truth* of any legitimate mathematical theory (or the existence of the objects of which it speaks). These further theses will be discussed shortly; I'll go on to argue that a view in the spirit of Cantor is correct, and is in fact the best way of accommodating Sociological Freedom.

Let us turn now to Sociological Freedom in contemporary mathematics. The natural place to look for evidence is in the contemporary study of what we might call

[2]For much more historical detail, see Kline [1972a].
[3]See Kline [1972b].
[4]Cantor [1996, Section 8].

*non-orthodox* mathematical theories. "Orthodoxy", here, as I understand it, is synonymous with the standard set-theoretic foundations for mathematics. It was a striking discovery in the foundations of mathematics in the twentieth century that it is possible to interpret the vast majority of contemporary mathematics within standard axiomatizations of set theory – typically Zermelo Fraenkel set theory plus the Axiom of Choice (ZFC) – and thus that, if we are willing to regard all mathematical objects as being (or as represented by) sets, virtually the whole of mathematics can be formalized in set-theoretic terms. As Enderton puts the point:

> Mathematical objects (such as numbers and differentiable functions) can be defined to be certain sets. And the theorems of mathematics (such as the fundamental theorem of calculus) then can be viewed as statements about sets. Furthermore, these theorems will be provable from our axioms [the axioms of ZFC]. Hence, our axioms provide a sufficient collection of assumptions for the development of the whole of mathematics—a remarkable fact.[5]

But I believe that Enderton's concluding claim here is too strong. Although it is certainly true that *the vast majority* of mathematics can be developed within the axioms of ZFC, it is far from clear that the *whole* of mathematics can, for there are many examples of legitimate mathematical investigation of systems that cannot be so easily shoehorned into a set-theoretic framework (or, at least, the framework generated by ZFC). These can be divided into two main classes: what we might call the *extra-orthodox* and what we might call the *un-orthodox*.

Take first consider *extra*-orthodox mathematics, in the sense of systems that *extend* but do not contradict orthodox foundations. For arguably (and somewhat ironically) the orthodox foundations of mathematics as codified by ZFC are insufficient to account for the practice of contemporary set theory itself. The reason for this is simple: the bulk of research in contemporary set theory concerns the investigation of large cardinal hypotheses, i.e. claims of the existence of various "large" levels of infinity that go beyond those that can be shown to exist within ZFC.[6] Cardinal hypotheses are often divided into "small" and "large" large cardinal hypotheses, the dividing line being between those that can and cannot hold in Gödel's constructible universe. At any rate, large cardinal hypotheses – at least the ones that arise in "mathematical nature", so to speak – can be arranged in a linear hierarchy in terms of degrees of interpretability: the crucial point for our purposes being that even the weakest among them, the existence of an inaccessible cardinal, takes us beyond ZFC.[7] But the practice of contemporary set

[5]Enderton [1977, p.10].

[6]A very nice overview can be found in Koellner [2011].

[7]An inaccessible cardinal $\kappa$ is a cardinal such that, for all $\lambda < k$, $2^{\lambda} < \kappa$. Any inaccessible cardinal

theory is, for the most part, the study of large cardinal axioms of this kind: their power, their implications, their relations to one another, their ability to decide claims (like the continuum hypothesis) that are demonstrably independent of ZFC, and so on. Thus, set theoretic practice cannot easily be accommodated by orthodox foundations, at least if we understand that strictly, i.e. going as far as ZFC but no further. Nevertheless, set theory is a serious and well-regarded branch of mathematics. I think this is hard to dispute, as a sociological matter; its practitioners are commonly employed by respectable mathematics departments, and articles and books in the field are published by prestigious mathematics journals and publishing-houses. Indeed, Paul Cohen's work in set theory (the proof that the continuum hypothesis is not provable within ZFC, and the method of forcing) has received the Fields Medal. There is no way to reasonably deny that set theory is a paradigm case of legimate mathematical investigation. To be clear: I do not wish to say that proponents of ZFC-foundations have *no* ways of accounting for such investigations; I will consider some in Section 2 below. Here I wish only to claim that there is a prima facie case that perfectly legitimate mathematics can be done in systems which, at first blush, go beyond ZFC.

Turn now to consider *un*orthodox mathematics, in the sense of systems that (at least on first appearances) directly *contradict* orthodox foundations. For various reasons – some philosophical and some mathematical – mathematicians have been interested in, and have chosen to study, axioms systems that result from replacing certain of the axioms of ZFC by their negations (or axioms that imply their negations). Take as an example the Axiom of Foundation, which states that all sets are well-founded, i.e. there are no sets with infinitely descending chains of set membership. Peter Aczel has extensively studied the system that results from replacing the Axiom of Foundation with what he calls the *Anti*-Foundation Axiom (an axiom which is formulated in graph-theoretic terms but which entails the existence of non-well-founded sets).[8] It turns out that the resulting system is mathematically interesting, and it has been argued that non-well-founded sets can be put to explanatory purposes outside of mathematics, for instance, in modelling certain communication systems as well as various "circular" phenomena such as the Liar paradox.[9] At any rate, my concern here is not with the fruitfulness of the theory. My point here is to observe that Aczel's work counts, if anything does, as legitimate mathematics by the sociological standards of the discipline. Aczel is employed by a respectable department of mathematics; his work, including his work on non-well founded sets, is published by prominent mathematics journals

is a model of ZFC, so ZFC + "there exists an inaccessible" proves $Con(\mathsf{ZFC})$. It follows that ZFC cannot interpret ZFC + "there exists an inaccessible", and that the existence of an inaccessible is not provable within ZFC.

[8] See Aczel [1988].

[9] See Barwise [1987].

and publishing houses. There is no way to reasonably deny the claim that the study of set theory with Anti-Foundation is a case of legitimate mathematical investigation.

As a final example of an unorthodox theory, consider Quine's New Foundations – a system proposed by Quine as an alternative foundation for mathematics. For our purposes, the relevance of New Foundations is that it is a theory of sets which entails the existence of a universal set $V$, i.e. such that $V = \{x|x = x\}$. In standard treatments of set theory, the non-existence of a universal set is of course ruled out by a version of Russell's paradox. Thus New Foundations appears to contradict orthodox treatments of set theory. Nevertheless, I submit, the study of New Foundations counts as legitimate mathematics by the sociological standards of the discipline. It has been (and continues to be) explored by mathematicians who are employed by respectable departments of mathematics; again, their work is published in prominent mathematics journals and publishing houses.[10] Although it is fair to say that the theory is not very widely adopted, I think that there is no way to reasonably deny the claim that the study of New Foundations is a case of a legitimate mathematical investigation.

There is a strong case, therefore, that Sociological Freedom is true. In turn, I believe that Sociological Freedom strongly motivates the following thesis:

**Epistemic Freedom:** Mathematicians can *justifiably* work within and draw out the consequences of any theory meeting minimal standards of logical coherence, in the sense that doing so manifests no epistemic error.

The link between the theses is straightforward. Mathematics is a paradigmatic case of a successful field of human inquiry, and a wildly successful one at that. Any plausible methodological stance will, accordingly, work under the standing presumption that its internal norms are justifiable, in any more general epistemic sense available to us. If that is right, then Epistemic Freedom quickly follows.[11]

## 2. In Search of a Rationalization

Let "Mathematical Freedom" denote the conjunction of Sociological and Epistemic Freedom. I have presented the beginnings of a case for Mathematical Freedom: the claim that, in effect, there are no restrictions, beyond certain fairly minimal logical ones, in the systems that mathematicians may legitimately study, both by the internal standards of their discipline and by more general epistemological standards. I want to do two things in this section: (i) motivate the thought that Mathematical Freedom poses a

[10]See for instance Holmes [1998] and Forster [1995].

[11]See Burgess and Rosen [1997] for an expression of a similar, if stronger, view. As they put it: "the search for 'real' standards exterior and superior and ulterior to those of common sense and the refined common sense which is science is pointless if not unintelligible."

*prima facie* philosophical challenge and (ii) explore and criticize various answers forthcoming from contemporary views in the philosophy of mathematics.

Why does Mathematical Freedom pose a *prima facie* philosophical challenge? Because if it is true, then it is a striking fact, and one that seems to distinguish mathematics from many other, if not all, serious forms of intellectual inquiry. A comparison with the empirical sciences may prove instructive. On all but the most revisionary views of, say, physics, there is a clear sense in which certain investigations can be *discredited* or *ruled* out as legitimate physical investigations on the basis of empirical observations – I take it that no-one who today studies for instance, phlogiston or phlogiston-theory is legitimately conducting a *physical* investigation.[12] But if Mathematical Freedom is true, then mathematical theorizing faces no such threat: a mathematical theory cannot be discredited or ruled out as illegitimate except by showing that it leads to contradiction. It would be too strong to say that Mathematical Freedom is a great *mystery* – in a moment I will look at reasons why mathematics might be so as to make it true. But it is nevertheless a striking fact, and one that accounts of mathematics need either to explain or explain away.

Many prevailing philosophical views of mathematics have in their own ways sought to account for Mathematical Freedom. There are three broad classes of response that can be discerned from the contemporary literature, each of which deserves discussion: revising mathematics ("revision"), restricting the range of legitimate mathematical theories ("restriction"), and by an appeal to ontology ("the cosmos").[13] I will take each in turn.

## 2.1 Revision

Firstly, a number of accounts hold out the prospect of explaining Mathematical Freedom by presenting a revisionary conception of mathematics. Revisionary in what sense? In that they deny the natural, pre-philosophically intuitive view that mathematics is a discipline that (i) issues in genuinely assertoric *claims*; which (ii) are to be

[12] No doubt such an investigation would have legitimately counted as physics at some point in history, and even today may be of concern to students of the history of physics. But on any remotely plausible view, according to which the goal of physics is the production of true or approximately true or empirically adequate theories about the physical world, the (now clear) falsity or empirical inadequacy of such a theory rules it out as a legitimate contender. Objection: aren't there false or empirically inadequate theories that are still nevertheless legitimately studied as part of physics, such as classical mechanics? Response: classical mechanics has relevant additional features. First, it plays a clearly valuable heuristic and pedagogical role in understanding and teaching its successor theories. Second, if any sense can be made of the notion of an "approximately true" theory, Newtonian mechanics is a paradigm case; see for instance the work of many structural realists, who argue that a great deal of structure is retained in the move from classical mechanics to its quantum mechanical successors.

[13] In a longer version of the paper, I also discuss Carnapian and more contemporary conventionalist views, but for reasons of space these are not treated here.

taken at *face-value*; and (iii) at least some of these claims are *true*.[14]

One broad family of such views are generalized *if-thenist* views, which deny the claim that mathematics ought to be taken at face value. On such views, mathematical claims are better understood in a hypothetical or conditional manner – the precise details of which require elaboration. For instance, a very crude version (discussed simply as "if-thenism" in the literature) will reinterpret "there exists a prime number" as a conditional whose antecedent is the conjunction of the axioms of arithemetic, and whose consequent is "there exists a prime number".[15] This view runs into a number of problems; a more sophisticated approach interprets mathematical claims using modal resources. For instance, on a view of Hellman's, "there exists a prime number" is reinterpreted, roughly, as the conjuction of a categorical claim – "an $\omega$-sequence [a model of arithmetic] is possible" and a hypothetical claim – "in any possible $\omega$-sequence, there exists a prime number" – (where the relevant notion of possibility is intended to be a broadly logical one).[16]

If such a view is correct, it would appear to provide a compelling explanation of Mathematical Freedom: for whenever a proof of some theorem is given (proceeding from a particular set of axioms), that will establish the truth of the relevant conditional or modal if-thenist claim. If the proper subject-matter of mathematics consists of conditional or hypothetical claims of this sort, then it is no mystery why mathematicians are free to construct proofs from any coherent set of axioms; for a successful and competently constructed proof from any set of axioms will demonstrate a claim of this kind. Note too that, at least on Hellman's version of the view, the "logical coherence" condition is also clarified and explained, in terms of the fact that any reinterpretation of a mathematical claim includes a categorical component stating outright that the relevant theory is logically possible.

Another influential family of revisionary views are broadly fictionalist in orientation; here I include not only Hartry Field's fictionalism but also Steven Yablo's figuralism and related views of Mary Leng.[17] Such views take mathematical statements not to be (when uttered in normal circumstances) assertoric, but something else altogether: offered in a fictional spirit ("according to the fiction of mathematics...") or figuratively (or otherwise non-literally). Such views typically go on to either deny or remain ag-

[14]A face-value reading of mathematics is just one at which our best theoretical reconstruction of mathematical claims is taken to match their surface form; see Benacerraf Benacerraf [1973].

[15]The view is often traced back to Russell [1903]: "Pure Mathematics is the class of all propositions of the form 'p implies q', where p and q are propositions containing one or more variables, the same in two propositions, and neither p nor q contains any constants except logical constants."

[16]Hellman [1989], inspired by Putnam [1967]. Only "roughly", because Hellman's view does not in fact appeal to metatheoretical notions like "a model of arithmetic", preferring to use second-order resources to simulate such talk. These details aren't really of relevance, though.

[17]See Field [1980], Yablo [2001] and Leng [2010].

nostic about the literal truth of mathematical claims. Views of this sort can plausibly account for Mathematical Freedom in a way directly analogous to the if-thenist: mathematical investigations are the exploration of fictional or figurative possibilities, and these are unconstrained (perhaps except minimally, by considerations of logical coherence).

The problem with revisionary views is simple. It is that the pre-theoretical understanding of mathematics according to which (i-iii) are true is one that holds considerable force. Mathematics is a subject that satisfies all of the hallmarks of a successful domain of inquiry. Any plausible methodological stance respects this, and treats proposals that understand the subject as failing to issue in genuine assertions, or considerably reinterpreted in ways that distort the face-value content of its claims, or regarded as systematically mistaken, as proposals which thereby inflict large costs. This is not to say that mathematics is somehow sacrosant or immune to philosophical challenges – of course it is not, and its presumptive good standing can at least potentially be rescinded. But it is to say that revisionary proposals should not be taken lightly. If it turned out that the only way to explain Mathematical Freedom was to embrace a revisionary proposal of the sort discussed above, then I think that would be a forceful consideration in favour of revisionism. But as I will soon go on to argue, there are better ways that do not ask us to bear this theoretical cost.

### 2.2 Restriction

Let us now turn to the class of views which attempt to *deny* Mathematical Freedom by arguing that, despite its initial plausibility, the freedom of mathematicians to investigate axiomatic structures is restricted. In order to be plausible, I take it that such views must have something to say about (i) where exactly the boundary between mathematically legitimate and non-legitimate theories lies, and (ii) an explanation of why such a boundary is in place.

The most influential kind of restrictionist view has been touched upon already: it is what I will call set theoretic fundamentalism, according to which the limits of legitimate mathematics are bounded by the limits of *set theory*.[18] On such a view, Mathematical Freedom will be false or heavily qualified. Nevertheless, set theoretic fundamentalism might be thought to vindicate a claim in the vicinity: that mathematicians are free to study any structure or system *as long as it can be interpreted within set theory in the right way*. Perhaps the most developed view of this kind is the one proposed by Penelope Maddy, according to which set theoretic practice "must strive to settle on one official

[18]My name for the view is slightly tongue-in-cheek; in my defense it is literally accurate, since the view's proponents hold that set theory is the ultimate foundation for mathematics.

theory of sets, a single, fundamental theory".[19] This is because set theory:

> hopes to provide a foundation for classical mathematics, that is, to provide a dependable and perspicuous mathematical theory that is ample enough to include (surrogates for) all the objects of classical mathematics and strong enough to imply all the classical theorems about them. In this way, set theory aims to provide a court of final appeal for claims of existence and proof in classical mathematics: the vague question 'is there a mathematical object of such-and-such description?' is replaced by the precise question 'is there a set like this?'; the vague question 'can so-and-so be proved mathematically?' is replaced by the precise question 'is there a proof from the axioms of set theory?' Thus set theory aims to provide a single arena in which the objects of classical mathematics are all included, where they can be compared and contrasted and manipulated and studied side-by-side.[20]

But there is a problem here. How can such a view account for the legitimacy of extra-orthodox and un-orthodox mathematical theories?

Take extra-orthodoxy first. Many set theoretic fundamentalists, Maddy included, would argue that they are in a perfectly good position to account for the practice of contemporary set theory. The legitimacy of investigating axioms extending $\mathsf{ZFC}$ could then be explained in terms of a desire to compare different possible developments of set theory so that the "correct" one can be adopted. Typically, philosophers sympathetic to such a view argue that – by analogy with scientific theory choice – the question of how to extend set theory can be resolved by employing general abductive methods, i.e. by comparing the different theories in terms of their possession of theoretical virtues like simplicity, fruitfulness, explanatoriness, and so on. Can this be correct? It certainly has implications that seem to conflict with Mathematical Freedom. For consider a possible scenario in which we have extended $\mathsf{ZFC}$ by adding a new axiom $\mathsf{A}$ that, say, decides against the continuum hypothesis, on (what the set theoretic fundamentalist would take to be) conclusive abductive grounds.[21] How should we regard the study of $\mathsf{ZFC} + \neg\mathsf{A}$? According to Mathematical Freedom, and, I claim, the dominant norms governing the practice of mathematics, it would be legitimate to study such a theory. But it is hard to see how a set theoretic fundamentalist view could agree here. If $\mathsf{A}$ really is conclusively supported on abductive grounds, presumably $\mathsf{ZFC} + \neg\mathsf{A}$ should be regarded as a discredited or refuted theory, as much in the same way that phlogiston theory is considered to be a discredited or refuted *physical* theory.

[19] Maddy [2007, p.354].

[20] Maddy [2007, p.354].

[21] The most systematic advocate of an approach of this kind is Hugh Woodin. See for instance Woodin [2001a,b].

One move that a set theoretical fundamentalist might make in response adverts to the completeness theorem. We know by the completeness theorem that any (proof-theoretically) consistent theory has a model, i.e. a set.[22] So, it might be argued, a proponent of $\mathsf{ZFC} + \mathsf{A}$ as the one true set theory could nevertheless account for the legitimacy of studying $\mathsf{ZFC} + \neg\mathsf{A}$: on such a view, it would amount to the study of models of that theory, where models are themselves taken to be entities described by $\mathsf{ZFC} + \mathsf{A}$. In this way, the "deviant" set theory might be accommodated within "orthodox" set theory. However, I do not think that this proposal is a satisfying one. It amounts, in effect, to advocating the systematic reinterpretation of claims made in the "deviant" set theory; a claim $\phi$ cannot be taken at face value, but becomes, presumably, something like "in any model of $\mathsf{ZFC} + \neg\mathsf{A}$, $\phi$ holds". But for reasons already mentioned, there is considerable pressure not to have to reinterpret mathematics in this way. At the very least, it is a costly endeavour, and one that should not be taken lightly.

The same issues recur when applied to unorthodox mathematics, say, Azcel's study of Anti-Foundation or New Foundations. One option would be simply to deny the legitimacy of the study of such theories altogether. But I believe, in light of the examples I have given in Section 1, that this is a deeply unpalatable response. Indeed, Maddy herself seems disinclined to take it, for at least in places she is sympathetic what she calls "the free pursuit of pure mathematics" – the freedom of mathematicians "to pursue their own goals, to investigate whichever mathematical structures they found interesting and fruitful."[23] Indeed, she thinks that set theory is placed to accomodate this: for "set theory is to serve as arbiter of mathematical existence", and so it should be "as generous as possible, so as not to constrain the free pursuit of pure mathematics".[24] But there is a serious tension in her views here. How can pure mathematics be both free and yet constrained by a particular mathematical theory, the theory of sets as codified by ZFC (perhaps supplemented with new axioms)? The dilemma is that Mathematical Freedom is *prima facie* incompatible with set theoretic fundamentalism. If the incompatibility is a real one, as I have argued, then I believe we should resolve the dilemma by giving up set theoretic fundamentalism. It is surely true that set theory is a powerful and elegant theory, and it is remarkable that it can be used to provide a foundation for so much of contemporary mathematics. But to insist that it is the *measure* of legitimate mathematics is to unacceptably limit the freedom of mathematicians in a way that cannot be justified, either theoretically or by appeal to mathematical practice.

[22]Nothing like the full strength of ZFC is required for the completeness theorem; it can in fact be carried out in a subsystem of second-order logic known as $\mathsf{WKL}_0$. See Simpson [1999, p.139].

[23]Maddy [2007, p.345].

[24]Maddy [2007, p.358].

## 2.3 The Cosmos

Let me turn to a final class of views: those which accept Mathematical Freedom and seek to explain its truth by appealing to premises of a metaphysical nature concerning the extent of the mathematical universe. The most salient view in this class is plenitudinous platonism as defended by Mark Balaguer.[25] In one regard, this view adopts a more or less standard set of platonist commitments: mathematics is the study of a distinctive kind of object, i.e. mathematical objects, which are abstract (i.e. non-spatio-temporally located, causally inert, etc) and exist mind independently; and mathematical claims are to be interpreted at face value and as being about such objects. But although typical – non-plenitudinous – platonists work with a picture of the mathematical universe in which there is, for example, a *single* realm of sets, and work with a conception of the mathematical project as that of correctly describing this universe, plenitudinous platonists by contrast hold that the mathematical realm is maximally plenitudinous in the sense that it includes an (intended) model of every logically consistent mathematical theory.

Balaguer himself presents his view primarily as an antidote to the epistemological worries plaguing other forms of platonism. But although he does not emphasize the point, it is more interesting for present purposes to note that the view arguably holds out the prospect of explaining Mathematical Freedom. The envisioned explanation is simple: if every consistent mathematical theory is true, then the investigation of any consistent set of axioms will satisfy the traditional platonist conception of mathematics as the study and description of a realm of mind-independent mathematical objects. So on the plenitudinous view, the freedom of mathematicians to investigate any (consistent) theory they desire will be explained and legitimated by the sheer vastness of the mathematical universe – a universe that is so vast, indeed, that it contains an intended model for any consistent theory we can come up with.

But is it ultimately a tenable response? I have two reservations about the view, which together serve to considerably undermine its attractiveness. Despite the reservations I will enter, I believe that there is something very much right in the vicinity of plenitudinous platonism. Indeed, in order to explain Mathematical Freedom, I will be arguing for a view that has roughly the same upshot: namely that every coherent theory is true of some intended model. But as we shall see, although my view comes to roughly the same conclusions as plenitudinous platonism, it does so in a very different way – one which, I will argue, is not vulnerable to the same objections.

[25] Balaguer [1998]. In fact Balaguer does not assert this view outright, but argues that it is the most plausible form of platonism; indeed, he goes on to argue that there is no fact of the matter in the dispute between platonists and anti-platonists. Nevertheless for the sake of simplicity I will continue to write as though Balaguer embraces plentitudinous platonism, and leave the qualifications to this footnote.

On to the reservations. The first is simply that the major metaphysical claim on which the view depends – the plenitude of the mathematical universe – is one which calls out for explanation, and yet no acceptable explanation appears forthcoming. It is no exaggeration to say that in Balaguer's book, half of which is dedicated to defending plenitudinous platonism, the only argument offered in favour of the claim of plenitude is that it successfully solves the epistemological challenge facing mathematics. Perhaps the most charitable reading is one on which he proposes acceptance of the plenitude claim as the result of an inference to the best explanation – perhaps something like: we have a great deal of mathematical knowledge; mathematics is to be given a face value reading; the best possible explanation of these facts would be the truth of plenitudinous platonism; therefore we should accept plenitudinous platonism. But even if that is mode of argument he employs, it is still nevertheless the case that plenitude is a striking fact, and one which requires an explanation.

It is of course open to Balaguer to claim that although the plenitude of the mathematical universe is striking, it is nevertheless a fact that might be reasonably taken as explanatorily basic. For instance, he might try to draw an analogy between the plenitude of the mathematical universe and, say, physical laws, arguing that facts of both kind are fundamental in the sense that they do not require an explanation. But I think that such a move would be deeply implausible. The claim of mathematical plenitutde is more than just a straightforward claim concerning mathematical existence: rather, it postulates a kind of massive correlation between two very different domains: on the one hand, facts about mathematical truth and existence, and on the other, *logical* (or even *syntactic*) facts concerning the consistency of various theories. It is hard to believe that such a correlation can be taken as explanatorily basic, because it seems *a priori* incredible that these two disparate domains would line up in the way claimed unless there is some deeper explanation. So until an explanation is forthcoming, plenitudinous platonism seems to rest on an unexplained – perhaps even inexplicable – mystery; and that is, at the very least, an aspect of the view that motivates consideration of alternatives.

My second reservation is broadly metasemantic in nature. Even supposing that we grant Balaguer's plentitudinous conception of the mathematical universe, there is still a mystery about how our theories could be about the relevant part of that universe. Slightly more precisely: the plenitudinous platonist owes an explanation of how it is that mathematical singular terms come to refer to the mathematical objects that they supposedly do, and similarly, how mathematical predicates come to denote the mathematical properties that they do. In one sense, this is just a traditional challenge for platonism reasserting itself: for just as platonism is often thought to face an epistemological challenge – how can we explain our knowledge of mathematics? – so too is it often though to face a referential or semantic challenge – how can we explain how mathematical language receives its semantic content (given that it involves mind-independent objects and their properties)?[26] Balaguer's preferred response is to invoke a distinction between what he calls "thick" and "thin" ways that a belief or theory can be about an object, where to "have a belief that is thickly about an object x, one must be connected to x in some appropriate way" – where a paradigm case of such a connection is causal – whereas thin beliefs, by contrast, require no such connection. Balaguer suggests that is enough for a platonist to explain how mathematical beliefs or theories are *thinly* about mathematical objects. And it emerges that Balaguer is thinking of the thin notion of aboutness in a very thin way indeed, for he suggests that a theory T is "thinly about" a plurality of objects if those objects satisfy T *under some reinterpretation of* T*'s quantifiers, predicates, and singular terms*. But this seems hopeless as a response, for a number of reasons. Firstly, the view will lead to massive referential indeterminacy, for it implies that a theory T will be "about" *every* model or plurality that satisfies it; at any rate, Balaguer seems willing to bite that bullet. But more importantly, this move seems to undermine the substantiality of the view altogether, for it undermines the distinction between intended and non-intended models of mathematical theories. To reassert this distinction, what would be needed is a modified account along the following lines: a theory T is "thinly about" a plurality of objects if those objects satisfy T under an *intended* interpretation of T's quantifiers, predicates, and singular terms. But to amend the account in this way is just to presuppose what is needed. An *intended* interpretation of a theory is just one where the objects of the interpretation are precisely those that are *genuinely referred to* by the terms of the theory and where the properties of the interpretation are precisely those that are *genuinely denoted* by the predicates of the theory; and these metasemantic relations are precisely what needed to be explained. After all, the entire challenge was to explain how mathematical terms refer or denote as they do, in the first place. So either we are still left wanting a metasemantic account for mathematics, or the plenitudinous platonist is forced to abandon the intended/non-intended distinction. The latter is hardly a plausible means of escape; without this distinction, plenitudinous platonism becomes the claim that every consistent theory has a model; but *this* is simply entailed by the completeness theorem for first order logic. On pain of trivialization, then, proponents of plenitude need a better metasemantic story than the one that Balaguer is offering.

[26]C.f. Burgess and Rosen: "[according to nominalists] there is a serious problem about how anything anyone says or does can establish a relationship of reference between words on our side of the great wall dividing the concrete from the abstract and entities on the other side." Burgess and Rosen [1997, p.51]. I don't think that one has to be a nominalist to motivate the idea that there's at least a *challenge* here – I take it that all sides will agree that the platonist owes a metasemantic account for mathematical language, i.e. an explanation of how mathematical terms get their semantic properties. In another context, this metasemantic challenge is discussed further in Warren and Waxman [forthcoming].

# 3. Alethic Pluralism and Coherence

So far, I've presented the thesis of Mathematical Freedom, and made a case that no satisfactory explanation is forthcoming from contemporary views in the philosophy of mathematics. Soon I will present my own solution. But before doing so, I need first to say a little about the doctrine of alethic pluralism, since my solution will draw upon it, and second to make some previously promised clarifications about the notion of logical coherence.

## 3.1 Alethic Pluralism

Alethic pluralism is a view about the nature of truth – thus, it is best viewed as a competitor to more well-known theories such as the correspondence theory, the coherence theory, pragmatist theories of truth, deflationism, and so on. In slogan form, alethic pluralism says that truth can be realized differently in different domains.[27] What is its appeal? The most common motivations stem from a dissatisfaction about traditional debates about the nature of truth – i.e. disputes over whether truth is correspondence, or coherence, or what a community of ideal enquirers would conclude, or whatever. Many philosophers find that the plausibility of these grand-scale theories of truth varies considerably according to the domain to which it is applied. For instance, a correspondence-based conception of truth – roughly, that the truth of sentences can be explained in terms of a metaphysically robust notion of "correspondence" with an independent realm of facts – is an attractive conception of truth when the sentence in question is about, say, physical objects, but deeply implausible when applied to claims about what is humorous or fashionable. And perhaps superassertibility – the property possessed by a claim when it is assertable relative to some body of information and never subsequently rendered unassertable – is a plausible account of truth when applied to humor or fashion, but is extremely unappealing as an account of the truth of claims about physical objects, which, intuitively, are wholly independent of the relevant standards of assertability.[28] Regardless of the success of these examples, there is surely some plausibility to the thought that the traditional answers to the nature of truth each overgeneralize.

So, alethic pluralists proceed on the basis that truth can be differently realized in different domains. While the discussion in the literature has tended to focus on the global merits of such a view as opposed to its competitors, the traditional theories of truth, it is less often remarked that it also holds out the prospect of resolving a number of domain-specific philosophical problems. The proposal in this paper is meant

[27] See Wright [1992], Pedersen and Wright [2013], and Lynch [2009].

[28] These proposals are taken from Wright [1992].

in precisely this spirit: I'll argue that alethic pluralism allows us to give a compelling explanation of Mathematical Freedom, one that is superior to the alternatives mooted above.

First, though, we need to talk about how the – so far, somewhat inchoate – thesis of alethic pluralism might be made more precise. A number of issues must be resolved.

(1) The target of the thesis. Is alethic pluralism a doctrine that concerns the *property* of truth, the *concept* of truth, the meaning of the word "*true*", or what?

(2) The nature of the *realization* relation. What is it to say that some property is the *local realizer* of truth within some restricted domain of discourse? Is realization to be understood as a form of metaphysical grounding or explanation? Or as satisfaction of an appropriate set of platitudes or functional role? Or some kind of constitution (e.g. the relation between determinates and determinables?)

(3) How is discourse to be divided into natural "regions" , within which truth can be differentially realized?

A full defense of alethic pluralism would need to provide satisfying and fully worked out answers to each of these questions. I will not pursue such a discussion here. While I'll be taking a version of alethic pluralism as a working hypothesis in what follows, in the process I will try to leave as much schematic as possible. So I will try to finesse these questions by appealing to (what will here be treated as a primitive) notion of realization, whose form is as follows: P (a property) locally realizes truth in domain D. Of course the notion of realization will presumably not be treated as primitive in the final analysis; but it will be helpful here to avoid committing myself to controversial answers to the questions above.

In order to make progress, however, I will need to say something about the criteria for taking a property to locally realize truth. Here I will follow a *platitude-based* strategy, whereby in order to make a case that some property locally realizes truth, it must be shown to satisfy a number of platitudes which encode principles that are commonly thought to govern the notion of truth.[29]

[29] There are many ways in which the platitude-based strategy might be fleshed out. For Wright, it arises from the project of giving a network analysis for the concept of truth – roughly, locating the target concept in a network of connections with other concepts in such a way as to provide illumination. For Lynch, truth is a functional kind – roughly, a property is a local realizer of truth if it plays "the truth role", the role or job that truth plays in regulating the discourse. A platitude-based strategy also emerges from such a view, where the platitudes are thought of as the specifiers of the relevant functional role. On reflection, "platitude" is probably a slightly infelicitous term in this connection, as it has connotations of obviousness and indefeasibility, but I follow the literature. Another question that ought to be addressed is: what exactly are the criteria that platitudes must satisfy in order to be considered as such? Proposals, none of which I want to commit myself to, include obviousness, folk acceptance, a priority, analyticity, and being (partially) constitutive of the concepts involved, to mention just a few. (See Wright [2013] and Lynch [2009] for further discussion.) While I don't want to deny that the issue needs to be addressed, I would note that it is an instance of a much more general problem facing many popular philosophical projects relying on Ramsey-Lewis-style methods of implicit definition for theoretical terms: by direct analogy, some criterion

What platitudes, in particular? As with any attempt to flesh out the functional role of a concept, there is no clear agreement. I am going to propose the following list of properties – taken from Wright and Lynch – as a plausible candidate. I will take it, then, that showing that some property locally possesses the following properties is enough to make a *prima facie* case for it as a local realizer of truth. Here are the platitudes:

i Transparency: for any $\phi$, "$\phi$" is true if and only if $\phi$;

ii Correspondence: for any $\phi$, "$\phi$" is true if and only if "$\phi$" corresponds to the facts;

iii Contrast: truth is to be contrasted with epistemic notions like justification (i.e. a claim can be true without being justified, and vice versa);

iv Opacity: particular truths may not be known in certain circumstances, or never be known, or unknowable in principle, and so on;

v Embedding: truth-aptness is preserved under a variety of operations – most saliently, logical operations like negation, conjunction, etc;

vi Timelessness: propositions are eternally true, if true at all;

vii Absoluteness: truth does not come in degrees and is not relative; propositions are absolutely true, if true at all.

viii Compositionality: truth obeys the standard compositional laws for the logical connectives. Thus, "$\neg\phi$" is true iff "$\phi$" is not true; "$\phi \wedge \psi$" is true iff "$\phi$" is true and "$\psi$" is true; "$\phi \vee \psi$" is true iff "$\phi$" is true or "$\psi$" is true; and so on.

### 3.2 Coherence

Various notions of "coherence" or "consistency" have cropped up in our discussion so far. There are several different ways in which such notions might be understood; it is time now to tidy things up.

The notions in the vicinity that first come to mind are mathematical ones: proof-theoretic consistency – the non-existence of a derivation (in some distinguished deductive system) of a sentence of the form $\phi \wedge \neg\phi$ – and model-theoretic consistency – the existence of a model satisfying the theory (where models are mathematical entities, i.e. sets). For my purposes, however, these notions will not do. The main reason is simple: in the account presented in this paper, I will wish to put a notion of coherence to work in explaining the nature of mathematical truth. But since both the proof- and model-theoretic notions are explicitly mathematical ones – defined in terms of mathematical

is needed for deciding what goes into the "theory" whose ramsification is supposed to characterize the term in question.

entities (sets, or derivations considered as abstract mathematical objects) – attempting to use them in this way will lead to a viciously circular account that will fail both on its own terms and in its ability to do the explanatory work desired.

Fortunately, there are independent reasons to pursue another approach. The basic idea, working within a tradition inaugurated by Kriesel and more recently elaborated by Field, is that we have recourse to an antecedently understood, primitive, logical notion, which I will henceforth call "logical coherence" or just plain "coherence".[30]

So, I shall proceed on the basis that we have in our language a primitive modal operator $\Diamond$, read as "it is (logically) coherent that..." , heuristically understood as something like truth in some logically possible scenario. [31] The dual notion, $\Box$, read "it is (logically) necessary that..." can be defined in the usual way ($\neg\Diamond\neg$). To say that this notion is to be taken as primitive is simply to say that its meaning is not given by defining it in terms of any other notion – most saliently, in terms of the model- or proof-theoretic notions of consistency. How *is* its meaning to be given? In terms of the procedural rules involved in using it, including in inferring with it, and its connection to other notions with which we are antecendently competent.

The first question concerns the modal laws are obeyed by the coherence operator on its intended interpretation. Some are fairly immediate: if $\phi$ is true in every logically possible scenario, then $\phi$ must be true; thus we have the (T) schema $\Box\phi \rightarrow \phi$. Similarly plausible is (K): $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$.[32]

[30]See Kreisel [1967] and Field [1989]. Kriesel originally presented his view as a way of coming to an adequate understanding of the completeness theorem for first-order logic. He argues that the philosophical implications of the theorem are best understood as showing that the proof- and model-theoretic notions are coextensive with an intuitive notion, by way of a famous squeezing argument. Think informally of coherence as truth in some logically possible scenario, and its dual notion, logical necessity or validity, as truth in all logically possible scenarios. Now suppose we have fixed upon an adequate formal deductive system for which we have soundness and completeness results. Soundness tells us that $\phi$ is a theorem of the system, then $\phi$ is intuitively valid. But on the proposed conception, this is equivalent to saying that $\phi$ is true in all possible structures; so, the thought goes, it must be true in all set theoretic *models*, in particular, because set theoretic models are just a special case of the more general intuitive notion of a possible structure. So if $\phi$ is intuitively valid then $\phi$ is model-theoretically valid. But by completeness, if $\phi$ is a first-order sentence, then if it is model-theoretically valid then it is a theorem. Kreisel's analysis of the situation is that, at least in the case of first-order sentences, these three notions – model-theoretic, proof-theoretic, and informal validity – can be seen to be coextensive in the first-order case. Field gives a couple of other compelling reasons why logical notions ought to be regarded as primitive, or at least not conceptually reducible to the mathematical notions. One is that it seems intuitively wrong to think that someone who e.g. denies an entailment claim is thereby committed to the existence of mathematical objects. Another is that there is no obvious reason why the model-theoretic account should be thought to be adequate in logics that go beyond first-order. The claim that every failure of implication is witnessed by a model is true in the first-order case, by appeal to the completeness theorem. But the analogous claim for *second-order* logic is equivalent to a fairly strong reflection principle. See Shapiro [1991] for more on the strength of second-order Kreisel's principle.

[31]As Field notes, it doesn't much matter whether we take as primitive a logical necessity *operator* $\Diamond$ or a logical implication *(binary) connective* $\rightarrowtail$, for each can be defined in terms of each other. $\phi \rightarrowtail \psi$ can be defined as $\Box(\phi \rightarrow \psi)$ (where $\rightarrow$ is the material conditional); and $\Box\phi$ can be defined as $(\phi \vee \neg\phi) \rightarrowtail \phi$.

[32]Iterated coherence claims seem prima facie intelligible, though for my purposes little turns on them.

Aside from the pure logic of the notion of coherence, a couple of additional bridge principles are worth mentioning. The intended interpretation supports the following principles:

> **Proof-Validity Principle** If there is a proof of $\phi$ in (an appropriate formal or informal deductive system S), then $\Box\phi$;
>
> **Model-Coherence Principle** If there exists a model of $\phi$, then $\Diamond\phi$.

The converse principles are worth considering. For explicitness they are:

> **Proof-Coherence Principle** If there is no proof of $\phi$ in (an appropriate formal or informal deductive system S), then $\Diamond\neg\phi$;
>
> **Model-Validity Principle** If there does not exist a model of $\phi$, then $\Box\neg\phi$

In the first-order case, these latter two principles can be shown to hold by Kriesel's squeezing argument. However, there is no reason to expect them to be true for sentences involving additional logical resources, e.g. stated in second-order logic, or first-order logic enriched with an operator "there exist finitely many..." In general, there is no reason why we cannot consider questions of logical coherence/validity when it comes to logics that extend first-order logic in this way (and hence which do not have complete proof procedures). Such logics sustain sound (but not complete) derivation systems, and possess a notion of "model", in such a way that the Proof-Validity Principle and Model-Coherence Principle will still come out as true. But the converse principles will in the general case be false, i.e. when completeness fails for the logic in question. So much for the set-up; we now return to the task of explaining Mathematical Freedom.

# 4. Mathematical Truth as Coherence

## 4.1 Hilbert's Dictum

There are two animating thoughts behind the proposal I am going to offer. The first is that what I will call *Hilbert's dictum*, taken from a reply to a letter from Frege in their

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Plausibly, if $\phi$ is logically necessary, it will be logically incoherent to deny that it is necessary, so we have $\Box\phi \rightarrow \neg\Diamond\neg\Box\phi$ which is equivalent to (4): $\Box\phi \rightarrow \Box\Box\phi$; thus coherence will obey at least S4. The status of the (5) axiom is slightly less clear. In his discussion of the issue, Field distinguishes two notions of logical necessity: a broad one, on which logical necessity is to be sharply distinguished from what is logically *knowable*, and a narrow one, akin to a kind of absolute provability, on which it is not. He points out that it is plausible to think that (5) holds of the broad notion, but not the narrow one (for even though no adequate proof procedure issues in all the correct claims of the form $\Diamond\phi$, on the broad conception – unlike the narrow one – that is no reason for denying that such claims are themselves logically necessary, i.e. that $\Diamond\phi \rightarrow \Box\Diamond\phi$).

notorious exchange, provides us with an extremely appealing explanation of Mathematical Freedom. Here it is:

> if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.

Although Hilbert's dictum is stated in terms of *contradiction*, suggesting that Hilbert was thinking in terms of the proof-theoretic notion, there is no reason why we cannot substitute our preferred notion of logical impossibility – a modification I will henceforth assume without comment.

The second animating thought is that Hilbert's dictum provides an extremely appealing explanation of Mathematical Freedom. It is possible to read the dictum as a statement of a kind of plenitudinous platonism, akin to the view of Balaguer's discussed above. For the reasons mentioned earlier, I believe that such a view has seriously unappealing features. There is, however, another reading, on which Hilbert's dictum is not to be understood as asserting a mere correlation, so to speak, between consistent mathematical theories and mathematical existence, but as one that primarily concerns the *nature of mathematical truth*. This is the reading that I will favour as we proceed. The connection to alethic pluralism should now be becoming clear: for, read in this way, Hilbert's dictum seems to require a background alethic pluralism for its plausibility. The dictum has considerable appeal taken as a thesis about mathematical truth and existence. But if it is taken as a *general* thesis about truth and existence, it is plaintly untenable – I take it that no-one would want to claim that every logically possible theory is true; this would have the absurd implication that e.g. there actually exist physical objects possessing any given attributes.[33] So we need some principled way of restricting the range of applicability of the dictum so that it applies to mathematics and only to mathematics. This is precisely where alethic pluralism comes into the picture: for if a view of mathematical truth can be made out which sustains Hilbert's dictum, then we will have found non-ad-hoc reasons to endorse a form of mathematical exceptionalism without committing ourselves to globally implausible consequences.[34]

All of these benefits, however, are merely prospective at this point. What we need, in order to show that they obtain, is a thesis that conforms to general alethic pluralist schema: truth in domain D is realized by property P. How should these schematic variables be filled out? Coming up with a suitable proposal is the task of this section.

[33] Even Lewisian modal realists will balk at the idea that any *logically* possible theory is true in some world.

[34] I believe that the view is also part of a larger class which allow for an extremely appealing solution to Benacerraf's famous dilemma; that, however, is a task for another paper, which I undertake in Waxman [MSa].

It will be helpful to consider how a couple of failed attempts need to be refined before arriving at a final proposal.

## 4.2 Truth as Coherence

Here, then, is a first pass. Consider the view (suggested by reading Hilbert's dictum somewhat literally), that

> **First Attempt** Truth in the mathematical domain is realized by the property of logical coherence.

There is a simple reason why this will not do: namely, it is straightforwardly contradictory. Consider an illustrative example, the Generalized Continuum Hypothesis: the claim that $\forall\lambda(Ord(\lambda) \to 2^{\aleph_\lambda} = \aleph_{\lambda+1})$ – for any ordinal $\lambda$, the next largest cardinal after $\aleph_\lambda$ is $2^{\aleph_\lambda}$. Now, by results of Godel and Cohen, if ZFC is formally consistent then so too are both ZFC + GCH and ZFC + ¬GCH. Let us suppose for the sake of argument that all of these theories are not only formally consistent but also logically coherent in the relevant sense, so that $\Diamond$ZFC, $\Diamond$ZFC + GCH and $\Diamond$ZFC + ¬GCH all hold. Then, according to this first pass proposal, we will obtain that both ZFC + GCH and ZFC + ¬GCH are *true* theories. But here a problem arises. If truth sustains the Equivalence Schema – "$\phi$" is true if and only if $\phi$ – as it had better, given that the schema is a core platitude, then we can immediately obtain the object-level contradiction GCH ∧ ¬GCH. So this first-pass attempt crashes pretty immediately. Thinking of truth as coherence *simpliciter* will not work when we are faced with theories that are (i) individually coherent yet (ii) jointly incoherent.

The problem here seems to stem from *the conflation of two different mathematical domains.* What I propose is that, instead of thinking of mathematics as a single, undifferentiated domain of discourse, with a single truth-realizing property, we think of it as constituted by a number of distinct domains – one for each concept of a type of mathematical object. The notion of a "concept of a type of mathematical object" will be made more precise as we go, but as a source of paradigm cases, think of the concept of natural numbers, the concept of real numbers, and the concept (or concepts, if one prefers) of sets. Then if we want to say that both ZFC + GCH and ZFC + ¬GCH are true (because they are both coherent), they must be understood as claims about different domains – if we allow for the sake of argument that the concept of set does not decide between GCH and its negation, these theories can be understood as being made about domains corresponding to two different concepts of set. To make this more precise, we must reformulate these theories so that their quantifiers are not, as in standard presentations of set theory, unrestricted; rather, we should introduce new predicates – say $Set_1$ and $Set_2$

– and rewrite the theories with their quantifiers restricted to these predicates. Thus, claims of the form $\forall x \phi$ become claims of the form $\forall x(Set_1(x) \rightarrow \phi)$ and claims of the form $\exists x \phi$ become claims of the form $\exists x(Set_1(x) \wedge \phi)$.

So the second proposal is this:

> **Second Attempt** every coherent theory $\mathsf{T}$ (formulated in the restrictedly quantified way above) characterizes a concept of a type of mathematical object and a domain of discourse $D_{\mathsf{T}}$. In each such domain $D_{\mathsf{T}}$, truth is realized by the property of following from the axioms of $\mathsf{T}$.

(where "following" is defined in terms of logical coherence, i.e. $\phi$ follows from the relevant axioms iff $\Box(Ax_{\mathsf{T}} \rightarrow \phi)$).[35]

This problem of conflation is thus solved: for instead of the (contradictory) $\mathsf{GCH} \wedge \neg\mathsf{GCH}$, we have that

$$\forall \lambda(Set_1(\lambda) \rightarrow (Ord_{Set1}(\lambda) \rightarrow 2^{\aleph_\lambda}_{Set1} = \aleph^{\lambda+1})$$

and also that

$$\neg\forall \lambda(Set_2(\lambda) \rightarrow (Ord_{Set2}(\lambda) \rightarrow 2^{\aleph_\lambda}_{Set2} = \aleph^{\lambda+1})$$

– and there is plainly no contradiction here.

But there is, though, a further, subtler problem. Recall that we require that the local realizer of truth within some domain satisfies the relevant platitudes governing truth. Well, among the least controversial such platitudes are compositional principles governing the logical connectives: for instance, "$\neg\phi$" is true if and only if "$\phi$" is not true, and "$\phi \vee \psi$" is true if and only if "$\phi$" is true or "$\psi$" is true. So any candidate to be a local truth-realizer had better sustain platitudes of this kind. But this is not the case on the present proposal, as can be seen by considering what we might call *incomplete* theories. Take $\mathsf{ZFC}$ as an example, and suppose that it is coherent. Then on the present proposal it thus gives rise to a concept of a kind of mathematical object and hence a domain of discourse, in which truth is realized by the property of following from the axioms of $\mathsf{ZFC}$. On very weak assumptions about logical necessity (and the previous assumption

[35] One complication that arises involves the use of "$Ax_{\mathsf{T}}$" where the theory $\mathsf{T}$ is infinitely axiomatized, as is $\mathsf{PA}$ and $\mathsf{ZFC}$. There are several ways that this might be handled. One is to assume recourse to (countable) infinitary conjunction, so that such conjunctions are formulable. However, this has the disadvantage of being far more powerful than is needed; after all, we do not need *arbitrary* countable infinite conjunctions, just certain ones with particularly syntactically uniform conjuncts. Another way to go would be to appeal to second-order resources, and restrict attention to theories that can be formulated using such resources (as $\mathsf{PA}$ and $\mathsf{ZFC}$ can), so that "$Ax_{\mathsf{T}}$" would come out as a finite conjunction. It might be thought that an appeal to second-order logic here would be overkill; but as far as I can tell, for the purposes of formulating claims like $Ax_{\mathsf{T}}$, nothing requires the second-order quantifiers here to be interpreted under the *full*, as opposed to the *Henkin*, semantics; and second-order logic under the Henkin semantics is mutually interpretable with a two-sorted first-order language. Another option still would be to allow "open ended" schemas in the object language, as opposed to in the metalanguage, as discussed in Field [2006], and then to allow such schemas to occur within the scope of the logical coherence operator.

that both $\mathsf{ZFC} + \mathsf{GCH}$ and $\mathsf{ZFC} + \neg\mathsf{GCH}$ are coherent), it can be shown that both the negation and disjunction principles fail. Take negation first: since $\Diamond(\mathsf{ZFC} + \mathsf{GCH})$, it follows that $\neg\Box(\mathsf{ZFC} \rightarrow \mathsf{GCH})$, and so that it is not the case that GCH is true. But since $\Diamond(\mathsf{ZFC} + \neg\mathsf{GCH})$, we also have $\neg\Box(\mathsf{ZFC} \rightarrow \neg\mathsf{GCH})$, so it is not the case that $\neg$GCH is true either; hence the distribution of truth over negation fails. Turning to disjunction: we have $\Box(\mathsf{GCH} \vee \neg\mathsf{GCH})$ and thus that $\Box(\mathsf{ZFC} \rightarrow (\mathsf{GCH} \vee \neg\mathsf{GCH}))$. But since $\mathsf{ZFC} + \mathsf{GCH}$ and $\mathsf{ZFC} + \neg\mathsf{GCH}$ are both coherent, we will have neither $\Box(\mathsf{ZFC} \rightarrow \mathsf{GCH})$ nor $\Box(\mathsf{ZFC} \rightarrow \neg\mathsf{GCH})$ , and hence that neither GCH nor $\neg$GCH is true; hence the distribution of truth over disjunction fails. These are serious problems, given the centrality of the relevant platitudes.

Turning to a diagnosis, the problem is that there is a sense in which ZFC is incomplete: it does not give us a maximally specified concept of a type of mathematical object, for it precisely does not "tell us" whether GCH or $\neg$GCH obtains with respect to the type of object it postulates. So we are led to say that *not* every coherent theory gives rise to a concept of a type of mathematical object – or at least, not to a maximally specified such concept – precisely because of such failures of distribution of truth over negation and disjunction. There are a number of ways that we might choose to proceed here. One would be to attempt to bite the bullet, and argue that mathematical truth does not, in general, distribute over negation or disjunction (at least, in domains whose concepts are not maximally specified in this way). Although such a view might be able to be worked out, it strikes me as philosophically unappealing and I will take it no further. Instead, my approach will be to stiplulate that the view applies only to theories that *do* fully specify the concept of a type of mathematical object. This corresponds to a formal property that we might call maximality: say that a theory T is maximal if it satisfies the following schema: $\Box(Ax_{\mathsf{T}} \rightarrow \phi) \vee \Box(Ax_{\mathsf{T}} \rightarrow \neg\phi)$. In other words, if T is maximal, then either $\phi$ is logically necessitated by the axioms of T, or $\neg\phi$ is.

So the final proposal then is this:

> **Final Attempt** Each coherent and maximal theory T characterizes a concept of a type of mathematical object and a domain of discourse $D_{\mathsf{T}}$, where truth in $D_{\mathsf{T}}$ is realized by the property of following from the axioms of T.

The assumption of maximality allows the distribution of truth over negation and disjunction to be restored. Negation is immediate. For disjunction, suppose for a contradiction $\Box(Ax_{\mathsf{T}} \rightarrow (\phi \vee \psi))$ but that $\neg\Box(Ax_{\mathsf{T}} \rightarrow \phi)$ and $\neg\Box(Ax_{\mathsf{T}} \rightarrow \psi)$. By maximality, $\Box(Ax_{\mathsf{T}} \rightarrow \neg\phi)$ and $\Box(Ax_{\mathsf{T}} \rightarrow \neg\psi)$, so $\Box(Ax_{\mathsf{T}} \rightarrow (\neg\phi \wedge \neg\psi))$ and hence $\Box(Ax_{\mathsf{T}} \rightarrow \neg(\phi \vee \psi))$ which contradicts the coherence of T.

One final worry, however, will need to be dealt with. It is that in conceiving of "domains of discourse" in the present way – as given by maximally specific concepts of types of mathematical object corresponding to *maximal* axiomatic theories – we have departed very radically from *actual* mathematical practice and what we might ordinarily think of as the domains of discourse within which *actual* mathematical assertions are made. This worry will be addressed in Section 5. There, I'll argue that when viewed in the proper light, this feature is actually a major virtue of the account, and sheds considerable light on the question of the determinacy of mathematical claims.

### 4.3 Checking the Platitudes

First, however, we need to confirm that the proposal above is defensible within an alethic pluralist framework, i.e. that it satisfies the relevant platitudes.

(i-ii) Transparency and Correspondence. I'll address these together. For the proposal to meet this platitude, it would need to be the case that, for all assertions $\phi$ in the domain $D_{\mathsf{T}}$ (concerning a type of mathematical object characterized by the theory $\mathsf{T}$), $\phi$ follows from $Ax_{\mathsf{T}}$ iff $\phi$ and $\phi$ follows from $Ax_{\mathsf{T}}$ iff $\phi$ corresponds to the facts.

I do not think it is possible to give a *straightforward* argument for these claims. My case for them is abductive: that accepting these claims leads to an overall philosophical account of mathematics that has significant advantages over its competitors, and thus that we should lend them credence. What advantages? One major advantage has been covered already: it is that this account explains Mathematical Freedom in a way that its competitors do not.[36]

Still, some support might be drawn from considering what it would take for these claims to *fail*. There are two possible sorts of counterexamples: from failures of the left-to-right direction of the biconditionals, and failures of the right-to-left direction. Take the first: suppose that, in some domain $D_{\mathsf{T}}$, we have an instance of a claim $\phi$ that follows from the axioms $Ax_{\mathsf{T}}$ but for which $\neg\phi$. The only reason I can think of for holding that there might be a claim of this kind is if it were held that axioms of $\mathsf{T}$ were *false* or otherwise misdescribed their domain. ("False" here can be understood in a merely disquotational sense, as a way of indirectly denying the claims). But, for the reasons discussed earlier, I believe that this possibility would make it very hard to explain Mathematical Freedom; given that thesis, it is difficult to accept that a mathematical theory can be false for any reason other than failing to be coherent.

Take the second possible kind of counterexample: suppose that $\phi$, but that $\phi$ doesn't follow from the axioms $Ax_{\mathsf{T}}$. Again, one motivation for thinking that there are propositions of this kind might be that the axioms are false, but this should be rejected for the reasons just given. Another might be that the axioms are true, but that they are not the whole story: truth in the relevant domain outstrips the axioms. However, given

[36]The case is bolstered further if, as I argue in Waxman [MSa], the view can also be shown to have significant epistemological advantages.

that the axioms are, by stipulation, maximal, this is untenable: if $\phi$ doesn't follow from the axioms, $\neg\phi$ does. It is therefore inconsistent to hold the triad consisting of: (i) $\phi$, (ii) the axioms themselves, and (iii) the claim that $\phi$ does not follow from the axioms. So, reflection on the shape of possible counterexamples give us considerable reason to endorse the thesis.

(iii) Contrast: Clearly, the property of being logically entailed by the relevant axioms is distinct from the property of being justified.

(iv) Opacity: Although it's controversial how much opacity is required of truth, the present proposal vindicates even the strongest formulations. It is possible for individual thinkers not to know certain facts about logical entailment; clearly, not all logical entailments are known by everyone to obtain. Similarly, some facts about logical entailment from axioms may never be known; for instance, facts about systems sufficiently uninteresting that no attention is ever paid to them, or facts about extremely complex entailments that lie beyond the capacity of humans to prove (as well as beyond the capacity of auxiliaries such as e.g. automated theorem provers) or otherwise epistemically support. Finally, it is at least possible that some facts about entailment are unknowable in principle (at least for beings like us); as would be the case if, for instance, it is essential to us that our cognitive powers are limited in various ways.

(v) Embedding: On the present proposal, a claim is truth-apt if the claim that it follows from the relevant axioms is truth-apt. Clearly, this property is preserved under all of the usual embeddings: e.g. if $\phi$ and $\psi$ are such claims, then so too is $\neg\phi$, $\phi \wedge \psi$, etc.

(vi) Timelessness: Facts about entailment are timeless.

(vii) Absoluteness: Facts about entailment are absolute.

(viii) Compositionality: This has already been demonstrated.

I conclude that there are good *prima facie* reasons for thinking that the proposal meets the requirements of alethic pluralism to be a truth predicate in good standing. In light of the overall attractions of the account, there is reason to proceed with optimism.

## 5. Determinacy in Mathematics

Let me return now to the problem mentioned above at the end of Section 4. On the present view, mathematics is divided into a number of domains of discourse $D_{\mathsf{T}}$, one for each coherent and maximal theory $\mathsf{T}$, where truth in the relevant domain is realized by the property of following from the axioms. But it might be thought that such a view seems hard-pressed to account for actual mathematical practice: when someone asserts or entertains Goldbach's Conjecture, the Mean Value Theorem, or the Generalized Continuum Hypthosis, in what sense can they really be said to be making an assertion of or entertaining a proposition in anything like a domain of discourse of this maximally specified sort? Since part of the motivation for the account in the first place was to explain Mathematical Freedom – a fact about mathematics as it is actually practiced – this question seems especially pressing. What is needed, therefore, is an explanation of how ordinary mathematical practice can be accomodated in terms of the proposal above.

My preferred explanation proceeds in two steps. The first part requires an (independently plausible) account of the metasemantics of mathematics. When we make ordinary mathematical claims, I take it that we do so against what I will call a background of "implicit axioms" – very roughly, commitments that a person or a community accepts or is disposed to accept as governing various bits of mathematical language. For instance, a claim like Goldbach's Conjecture incorporates *arithmetical* language, the implicit axioms of which (for anyone with a non-deviant view of arithmetic) will include the Peano axioms. The role of these implicit axioms is *metasemantic* in that they explain why arithmetical language has the content that it does.[37] The basic idea is this: the domain of discourse in which a particular mathematical claim inhabits depends, in the first instance, upon the implicit axioms that govern the vocabulary featuring within it.

So, if ordinary mathematical assertions did turn out to employ vocabulary governed by implicit axioms that are maximal in the sense discussed earlier, then there would be no problem here to speak of. But the second part of my explanation is motivated by the observation that they might easily fail to be so. For instance, suppose for the sake of argument that the axioms of ZFC exhaust the implicit axioms of intuitive set theory. Then, if both $\mathsf{ZFC} + \mathsf{GCH}$ and $\mathsf{ZFC} + \neg\mathsf{GCH}$ are coherent theories, it will follow that intuitive set theory fails to imply either GCH or its negation. My preferred diagnosis of such a case is one of indeterminacy: primarily, I think that this would mean that there is a kind of indeterminacy concerning *which mathematical domain* the claim GCH is made in. If this is right, then various other kinds of downstream indeterminacy will follow: indeterminacy in our set theoretic vocabulary, and in the claim GCH itself.

Far from being an unwelcome consequence of the overall view, I think that this account of implicit axioms and determinacy has considerable independent plausibility and significantly clarifies a number of philosophical debates. Let us take a fairly recalcitrant case: philosophical and mathematical disagreement over the adoption of new axioms for set theory (i.e. the issue of whether, and if so, on what grounds, such axioms can or should be accepted). Naively, three kinds of questions might arise:

[37]For familiar externalist reasons it might be that they are not explicitly believed, entertained, or even formulable using the conceptual capacities of a particular participant in the practice; for instance, we might attribute an implicit acceptance of the axiom schema of induction to anyone who is disposed to reason in the appropriate way, without requiring them to be able to write down the schema in first-order logic or the like.

- Objective questions: which implicit axioms are true or objectively correct?
- Descriptive questions: which implicit axioms *actually* govern particular mathematical vocabulary or concepts?
- Normative questions: which implicit axioms *ought* to govern particular mathematical vocabulary?

Disputes about the adoption of new axioms are sometimes understood as objective questions. Here my view takes a firm stance: any coherent set of axioms is as objectively correct as any other, in the sense that adopting those axioms would ensure that one's assertions (using the relevant vocabulary) are made in the domain in which those axioms are true. This is admittedly a somewhat deflationary response. But I believe that much of the dispute about set theory can be fruitfully recast in terms of the other two kinds of questions, even if there is substantial unclarity as to how this might go.

First, consider those who argue that reflection principles are implicit in our conception of set and can be used to motivate the addition of new axioms.[38] If such a move – arguing that reflection principles are implicit in our conception of set – is supposed to be dialectically relevant, as it surely is intended to be, it must be because it is the principles governing our *actual* set theoretic vocabulary that are in question, and so the question being addressed is (at least partly) a descriptive one. Second, consider the view that we should appeal to abductive or "external" considerations, i.e. considerations of the theoretical strength, explanatory power, and fruitfulness of resulting theories, to settle the question of new axioms. Some proponents of this view might take these considerations to be relevant to an objective dispute; but on my proposal this would be misguided for reasons already established, because questions of the objective correctness of axioms are wholly settled by the coherence of those axioms. A more promising reconstruction is to understand such a view as an answer to a normative question: a proposal about which new axioms ought to be adopted, not (as with descriptive disputes) as a means for making explicit what was already implicit in our concepts, but as a normative proposal intended to improve our set theoretic practice along salient dimensions of theoretical virtue. Third, it is also possible that abductive considerations could be relevant to descriptive questions. After all, there is no reason that we are forced to understand the implicit axioms of set theory narrowly, as stemming solely from our conception of the subject-matter; perhaps it could be argued that external considerations of theoretical virtue also discipline our use of the relevant vocabulary, so that (for instance) it is part of our implicit set theoretic practice that we are

[38] See for instance Tait [2005, chapter 6].

disposed to adopt axioms that score highly along the relevant dimensions of virtue.[39] A last point: for any given axiom, there is no *a priori* reason why descriptive and normative disputes must deliver the same verdict. For instance, it seems perfectly reasonable to think that the Axiom of Foundation is one of the implicit axioms of intuitive set theory, but nevertheless that it adds very little to the overall theoretical virtue of the theory.[40]

My view also allows for disputes about the determinacy of mathematics to be formulated in an attractive way, displaying how they might eventually be resolved. Take the question of whether, say, our arithmetical language is determinate. According to my view, this is equivalent to the question of whether the implicit axioms of intuitive arithmetic are maximal. This will turn on two further questions: (a) what are the implicit axioms of intuitive arithemetic? and (b) how "strong" is logical entailment? I take it that virtually everyone will agree that the first-order Peano axioms are implicit in intuitive arithmetic. The interesting question is whether it contains any additional implicit principles. Many popular approaches in the literature can be read as claiming that it does. For example, some authors have argued that intuitive arithmetic is better captured by a *second-order* formalization of the Peano axioms.[41] Others have argued that it is better captured in terms of *schematic* principles, where schemas are taken as primitive logical ideology not reducible to second-order sentences or sets of first-order sentences.[42] In addition, regarding (a), there is a considerable literature on what we might take to be "computational" constraints on the implicit axioms that govern arithmetic (and that apply generally); for instance, there are some who argue that constraints on human cognition imply that the implicit axioms must be recursively enumerable (and others who argue from determinacy to the *lack* of such computational constraints).[43] Regarding (b): views that incorporate logical resources that go beyond first-order logic (such as the schematic and second-order views mentioned above) will presumably accept a stronger than first-order notion logical entailment (in the "modal" sense we have been working with).

To reiterate, my aim here is not to resolve issues of the adoption of new axioms or of mathematical determinacy. I will be happy if I have provided a view which sets them in an attractive framework and which renders the core issues clearer than before.

[39] Such a view would, at least in one respect, collapse the dichotomy between "intrinsic" and "extrinsic" justifications of axioms.

[40] As Kunen [2011, p.47] puts it, "The Axiom of Foundation is, as always in mathematics, totally irrelevant."

[41] For instance, Kreisel [1967]: "A moment's reflection shows that the evidence of the first-order schema derives from the second-order [axiom]." See also Shapiro [1991] and Isaacson [2011].

[42] See Parsons [2008] and Lavine [1998] for influential presentations of such a view.

[43] The most (in)famous example of such a view is Lucas [1961]. See also Warren and Waxman [forthcoming] for discussion.

Instead of fruitless debates about the objective correctness of one set of axioms or another, we can focus on the real issues: what are we committed to, and what ought we be committed to, in the use of various kinds of mathematical vocabulary? Thus the apparatus brought in in this section to connect up my view of mathematical truth with the actual practice of mathematics, far from being an epicycle, instead allows us to make theoretical progress in a number of thorny disputes.

## 6. Conclusion

Let me briefly sum up what I have tried to achieve in this paper. I've introduced and motivated the thesis of Mathematical Freedom, and argued that it is not successfully accounted for by any of the dominant approaches in the philosophy of mathematics today. Against a background of alethic pluralism, I've advanced a distinctive proposal about how truth is realized in mathematical domains, which, I've claimed, explains Mathematical Freedom and has many other attractive features. Finally, I've sketched how disputes about new axioms mathematical determinacy might be understood and resolved in terms of my proposal.

Although a lot has been discussed, there are plenty of directions for future work. One issue concerns the ability of the view presented here to overcome Benacerraf's famous dilemma. On its first horn, a semantic theory for mathematics must cohere with our best semantic theory for our language as a whole, a constraint which militates in favour of platonism; on the other horn, a reasonable epistemology must be provided for mathematics, but this seems difficult if platonism is true, given our lack of access to mathematical objects. I believe, although it would require a paper of its own to defend these claims in full, that the view explored here provides, almost uniquely, the resources to overcome this dilemma.[44] Very roughly indeed: the view outlined here is indeed a form of platonism, since it holds that coherent mathematical theories are true and therefore that the objects described by them genuinely exist, are mind independent, etc. However, the particular account of mathematical truth given here appeals fundamentally only to the notion of coherence. And while it is a very good question what epistemic grounds we have for thinking that our best mathematical theories are coherent in the relevant sense – a question that also deserves sustained investigation, which I undertake in other work – there is every reason to hope that it is far more tractable than the epistemic problems faced by more traditional forms of platonism.[45]

[44] A full defence is undertaken in Waxman [MSa].

[45] Various aspects of the epistemology of coherence and consistency are discussed in Waxman [MSb], Waxman [MSc], and Waxman [MSd].

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